1.2 Natural numbers & divisibility test

1. Natural numbers

- $\mathbb{N} = \{0, 1, 2, 3, ...\}$ – the set of natural numbers
- $\mathbb{N}_+ = \{1, 2, 3, ...\}$ – the set of natural positive numbers
- **Prime numbers:** A number $n \in \mathbb{N}$, $n > 1$ is called **prime** if its only factors are 1 and itself.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

Theorem 1 (Euclid's theorem) *There are infinitely many prime numbers.*

Proof:
Consider any finite list of prime numbers $p_1, p_2, \ldots, p_n$. It will be shown that at least one additional prime number do not belong to that list. Let P be a product of all the prime numbers in the list: $P = p_1 \times p_2 \times \ldots \times p_n$. Let $Q = P + 1$. Then Q is either prime or not. If Q is prime, then there is at least one more prime that is not in the list. If Q is not prime, then some prime factor p_i, $(i = 1, 2, \ldots, n)$ divides Q. If this factor p_i were in our list, then it would divide P (since P is the product of every number in the list); but P divides $P + 1 = Q$. If p_i divides P and Q, then p_i would have to divide the difference of two numbers, which is $(P + 1) - P$ or just 1. Since no prime number divides 1, p_i cannot be on the list. It means that at least one more prime number exists beyond those in the list. This proves that for every finite list of prime numbers there is a prime number not in the list, and therefore there must be infinitely many prime numbers. *q.e.d.*

- **Composite numbers:** A number is called composite if it is not prime, that is if it has more than two distinct factors.

  Theorem 2 (Unique Factorization Theorem) *Every natural number greater than 1 is a power of a prime or can be expressed as a product of powers of primes. This factorization is unique, apart from the ordering. That is, If we ignore the order in which we write the prime factors there is only one prime factorization of every natural number.*

- **GCD(a,b)** - The Greatest Common Devisor
- **LCM(a,b)** - The Lowest Common Multiple

2. Divisibility of natural numbers

- **Even numbers** – divisible by two – $0, 2, 4, 6, 8, 10, 12, \ldots$
- **Odd numbers** – not divisible by two – $1, 3, 5, 7, 9, 11, \ldots$
- **Divisibility Rules**
  - `Divisibility by 0` — No numbers are divisible by 0
  - `Divisibility by 1` — All numbers are divisible by 1.

– `Divisibility by 2` — Even numbers are divisible by 2.

  Example: 109850 is divisible by 2 because it is an even number.

– `Divisibility by 3` — Add the digits of a number together. If the sum is divisible by 3, then the original number is divisible by 3.

  Example: The number 792 is divisible by 3 because $7 + 9 + 2 = 18$, and 18 is divisible by 3.

– `Divisibility by 4` — If the last two digits of a number are divisible by 4, then the original number is divisible by 4.

  Example: The number 16248 is divisible by 4 because the last two digits, 48, are divisible by 4.

– `Divisibility by 5` — If a number ends in 0 or 5, then the number is divisible by 5.

  Example: The number 563,021,689,540 is divisible by 5 because it ends in 0.

– `Divisibility by 6` — If a number is divisible by 2 and 3, then it is also divisible by 6.
  Example: The number 6874 is not divisible by 6, even though 6874 is even, indicating divisibility by 2, but $6+8+7+4 = 25$, and 25 is not divisible by 3.

– `Divisibility by 7` — Double the last digit and then subtract it from the number formed by the remaining digits.If the result is divisible by 7 or equal to 0, then the original number is divisible by 7. This can be repeated if necessary.

  Example: The number 3416 is divisible by 7 because: Double the last digit $6 \times 2 = 12$. Subtract the result from remaining digits $341 - 12 = 329$ Repeat if necessary with the result. In this case 329. $9 \times 2 = 18$, then $32 - 18 = 14$, and 14 is divisible by 7.

– `Divisibility by 8` — If the last three digits of a number are divisible by 8, then the original number is divisible by 8.

  Example: The number 5128 is divisible by 8 because $128 \div 8 = 16$, and 16 is divisible by 8.

– `Divisibility by 9` — Add the digits of a number together. If the sum is divisible by 9, then the original number is divisible by 9.

  Example: The number 65762 is not divisible by 9 because $6+5+7+6+2 = 26$, and 26 is not divisible by 9.

– `Divisibility by 10` — If the number ends in 0, then it is divisible by 10.

  Example: The number 29581940 is divisible by 10 because the last digit is a 0.

– `Divisibility by 11` — Alternately add and subtract the digits of the number. If the result is divisible by 11 or equal to 0 then the original number is divisible by 11.

  Example: The number 3564 is divisible by 11 because $3 - 5 + 6 - 4 = 0$.

– `Divisibility by 12` — If a number is divisible by 3 and 4, then it is also divisible by 12.

  Example: The number 409536 is divisible by 12 because $4+0+9+5+3+6 = 27$ which shows divisibility by 3, and the last two digits, 36, indicate divisibility 4.

EXERCISES

1. Use the divisibility rules to underline the answers. (*First line is made as an example*)

Number	**Divisible by**										
35	2	3	4	**5**	6	**7**	8	9	10	11	12
63	2	3	4	5	6	7	8	9	10	11	12
84	2	3	4	5	6	7	8	9	10	11	12
124	2	3	4	5	6	7	8	9	10	11	12
168	2	3	4	5	6	7	8	9	10	11	12
201	2	3	4	5	6	7	8	9	10	11	12
351	2	3	4	5	6	7	8	9	10	11	12
442	2	3	4	5	6	7	8	9	10	11	12

2. Factorize the following numbers:

   (a) 387

   (b) 452

   (c) 6400

3. Find the LCD(a, b) and the LCM(a, b) if:

   (a) $a = 12$, $b = 14$,

   (b) $a = 26$, $b = 68$,

   (c) $a = 40$, $b = 30$,

   (d) $a = 18$, $b = 141$

   (e) $a = 200$, $b = 120$

4. Find the LCD(a, b) and the LCM(a, b) using Euclidean Algorithm, if:

   (a) $a = 1625$, $b = 2565$,

   (b) $a = 2600$, $b = 6800$,

   (c) $a = 441$, $b = 1770$,

   (d) $a = 135$, $b = 620$

   (e) $a = 1464$, $b = 1204$